Evolution operators for describing linearly singular systems

Ángel Martínez Muñoz

UNIVERSITAT ROVIRA i VIRGILI

Joint work with Xavier Gràcia and Xavier Rivas

2nd International Workshop on Geometric Field Theory

July 6th, 2026

Table of contents

1 Introduction

2 Differential relations

3 Linearly singular systems

4 Examples

5 Linearly singular systems as vector fields along maps

6 Example: The evolution operator for classical mechanics

7 Conclusions

8 References

Introduction

The Picard–Lindelöf theorem guarantees the **existence and uniqueness** of solutions for an ODE in normal form:

$$\dot{x}^i = f^i(x)\,,$$

provided that the functions f^i are sufficiently regular.

Geometrically, these systems correspond to **vector fields**:

- If X is a v.f., in local coordinates it can be written as:

$$X = \sum_{i=1}^{n} f^i(x) \frac{\partial}{\partial x^i}\,.$$

- The integral paths of X, i.e., paths $\gamma\colon I \to M$ satisfying $\dot{\gamma} = X \circ \gamma$, are precisely the solutions of the ODE $\dot{x}^i = f^i(x)$.

Introduction

However, not all ODEs can be written in normal form ($\dot{x} = f(x)$).

- **Non-holonomic mechanics**.
- **Singular Lagrangians**.
- **Electrical circuit design** (differential-algebraic equations / DAEs).
- **Control of interconnected systems**.

Example (Euler–Lagrange equations)

The Euler–Lagrange equations for a Lagrangian $L(q, \dot{q})$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad \Longleftrightarrow \quad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j \,.$$

can only be written in normal form if the Lagrangian is **regular** (i.e., if we can invert the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$).

Differential relations

A general autonomous first-order ODE

$$f^i(x, \dot{x}) = 0$$

can be seen, geometrically, as a subset (or submanifold) $D \subseteq \mathrm{T}M$.
A path $\xi \colon I \to M$ is a **solution** if its tangent lift lies entirely within D:

$$\dot{\xi}(I) \subset D\,.$$

Remark

If $D \subset \mathrm{T}M$ is a submanifold, then we can regard it as being implicitly defined, locally, as the zero level set of some functions $\{f^i = 0\}$. Thus, locally, a path $\big(x(t), \dot{x}(t)\big)$ is a solution of $D \subset \mathrm{T}M$ iff it solves the ODE

$$f^i(x, \dot{x}) = 0\,.$$

If the ODE is in normal form, $D = X(M)$ for some v.f. $X \in \mathfrak{X}(M)$.

Differential relations

Definition

We say that a subset $D \subseteq \mathrm{T}M$ is a **differential relation**.

For a general differential relation, we **might not have solutions** for every initial condition, and even when they exist they **might not be unique**.

Example

Let $M = \mathbb{R}$ and define the implicit differential system

$$D = \left\{(x, v) \in \mathrm{T}\mathbb{R} \mid v^2 = x\right\}.$$

For the initial condition $p_0 = 0 \in D$ the constant path $\xi_1(t) = 0$ and $\xi_2(t) = \frac{1}{4}t^2$ are solutions, so there is no uniqueness of solutions. However, there is no solution for any initial condition $p = x$ with $x < 0$.

Constraint algorithm

In order to study the existence of solutions in this more general setting, one develops the **constraint algorithm**. We want to find $D' \subset \mathrm{T}M$ s.t.:

1. It is *equivalent* to the original differential relation, so D and D' have the **same solution paths**, and
2. There exists a solution through every initial condition $p_0 \in \tau_M(D')$.

Constraint algorithm

Any solution $\xi(t)$ of $D \subset \mathrm{T}M$ must lie in the projected base space and remain tangent to it at all times. This leads to the iterative method:

$$M_{i+1} \coloneqq \tau_M(D_i)\,, \qquad D_{i+1} \coloneqq D_i \cap \mathrm{T}M_{i+1}\,,$$

starting with $D_0 = D$.

At each step, we have to assume that M_{i+1} is a closed submanifold, and the algorithm will finish at a **final constraint submanifold** M_f.

Linearly singular systems: Initial definitions

We restrict our analysis to a more tractable subcase that models most physical systems of interest (e.g., the Euler–Lagrange equations), where the implicit system of ODEs is of the form

$$A(x)\dot{x} = b(x)\,,$$

with A a, generally singular, matrix and b a vector. Geometrically,

Definition (Linearly singular system)

Let M be a manifold and $\pi\colon F \to M$ a vector bundle.
A **linearly singular system** (or lss) is a pair (A, b) where:

- $A\colon \mathrm{T}M \to F$ is a morphism of vector M-bundles,
- $b\colon M \to F$ is a section of π.

A path $\xi\colon I \to M$ is a **solution** of (A, b) if we have $A \circ \dot{\xi} = b \circ \xi$.

Linearly singular systems: Initial definitions

In terms of commutative diagrams,

$$
\begin{tikzcd}
\mathrm{T}M \arrow[r, "A"] \arrow[d, "\tau_M"'] & F \arrow[dl, shift right, "\pi"'] \\
M \arrow[ur, shift right, "b"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathrm{T}M \arrow[r, "A"] \arrow[d, "\tau_M"'] & F \arrow[dl, shift right, "\pi"'] \\
I \arrow[ur, "\dot{\xi}"] \arrow[r, "\xi"'] & M \arrow[ur, shift right, "b"'] &
\end{tikzcd}
$$

In terms of differential relations, a linearly singular system defines the subset

$$D = A^{-1}(b(M)) \subset \mathrm{T}M \, .$$

Note that this might not be a submanifold.

Explicit realization of the constraint algorithm

Consider a linearly singular system (A, b). We want to find the set of points for which we have existence of solutions.
Since A is not an isomorphism in general, solutions $X(x) \in \mathrm{T}_x M$ of

$$A_x \cdot X(x) = b(x)$$

may not exist at every point $x \in M$.

Consistency condition: Solutions can only exist where $b(x)$ has a valid preimage. This defines the *first constraint submanifold*:

$$M_1 := \{x \in M \mid b(x) \in \mathrm{Im}\, A_x\}$$

If A has locally constant rank,

$$b(x) \in \mathrm{Im}\, A_x \qquad \Longleftrightarrow \qquad b(x) \in (\mathrm{Ker}\ {}^t A_x)^\circ$$

Explicit realization of the constraint algorithm

So, if $\{s^i\}$ is a local frame of $\mathrm{Ker}\ {}^t A$, then M_1 is described by

$$\phi^i = \langle s^i, b \rangle \qquad \text{and} \qquad M_1 = \{\phi^i = 0\} \,.$$

Definition

A function ϕ is a **constraint function** if $\phi \circ \xi = 0$ for any solution path ξ.

The constraint algorithm then continues (assuming constant rank and closed submanifold conditions) by applying the tangency condition

$$(X \cdot \phi^i) = 0 \,,$$

where X is a vector field s.t. $A \circ X = b$ is satisfied at points of M_1.
This either yields new constraints or fixes some degrees of freedom.
Applying this iteratively gives the constraints defining the final constraint submanifold.

Example: Hamiltonian systems

Definition

A **Hamiltonian system** is a triple (M, ω, H), where M is a manifold, $\omega \in \Omega^2(M)$ is a closed 2-form of constant rank and H is a function.

The solutions to Hamilton's equations are

$$\widehat{\omega} \circ \dot{\xi} = \mathrm{d}H \circ \xi \,,$$

where $\widehat{\omega} \colon \mathrm{T}M \to \mathrm{T}^*M$ is the vector bundle morphism induced by ω. This is also the equation of motion of the linearly singular system $(\widehat{\omega}, \mathrm{d}H)$

$$\begin{tikzcd}
\mathrm{T}M \arrow[r, "\widehat{\omega}"] \arrow[d, "\tau_M"'] & \mathrm{T}^*M \arrow[dl, shift right, "\pi_M"'] \\
M \arrow[ur, shift right, "\mathrm{d}H"'] &
\end{tikzcd}$$

Hamiltonian systems appear naturally in the study of the singular Lagrangians in classical mechanics. Geometrically, they were first studied by Gotay, Nester and Hinds in 1978.

Example: Dirac structures

A Dirac structure is a vector subbundle $L \subseteq \mathrm{T}M \oplus \mathrm{T}^*M$ s.t. $L = L^\perp$,

$$L^\perp = \{(Y,\beta) \in \mathrm{T}M \oplus \mathrm{T}^*M \mid \langle \alpha, Y\rangle + \langle \beta, X\rangle = 0 \text{ for all } (X,\alpha) \in L\}\,.$$

The dynamics associated with a Dirac structure are implicit Hamiltonian systems, i.e. given a function H we look for the vectors X such that

$$(X, \mathrm{d}H) \in L$$

In terms of linearly singular systems, the integral paths of the vector fields X are the solutions to the linearly singular system

$$\begin{tikzcd} \mathrm{T}M \arrow[r, "\iota"] \arrow[d] & (\mathrm{T}M \oplus \mathrm{T}^*M)/L \\ M \arrow[ur, "{[(0,-\mathrm{d}H)]}"'] & \end{tikzcd}$$

Example: Vector fields along maps

A **vector field along a map** $\alpha\colon N \to M$ is a map $K\colon N \to \mathrm{T}M$ such that:

$$\tau_M \circ K = \alpha\,.$$

Equivalently, a vector field along a map α is a section of the pullback bundle $N \times_\alpha \mathrm{T}M$. With this, a vector field along a map defines a lss

$$\begin{tikzcd}
\mathrm{T}N \arrow[r, "\mathring{\mathrm{T}}\alpha"] \arrow[d, "\tau_N"'] \arrow[rr, bend left, "\mathrm{T}\alpha"] & N \times_\alpha \mathrm{T}M \arrow[r, "\pi_2"] \arrow[dl, shift right, "\pi_1"'] & \mathrm{T}M \arrow[d, "\tau_M"] \\
N \arrow[ur, shift right, "K'"'] \arrow[urr, "K"'] \arrow[rr, "\alpha"'] & & M
\end{tikzcd}$$

where the maps are $\mathring{\mathrm{T}}\alpha = (\tau_N, \mathrm{T}\alpha)$ and $K' = (\mathrm{Id}_N, K)$.

Examples: Vector fields along maps

- The solutions of this linearly singular system are given by the paths $\xi \colon I \to N$ such that
$$\mathrm{T}\alpha \circ \dot{\xi} = K \circ \xi \,.$$
- If the path ξ is a solution of the system then, for any function $h \in \mathscr{C}^\infty(M)$, we have
$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha^*(h) \circ \xi) = (K \cdot h) \circ \xi \,.$$
- Any vector field along a map naturally defines another differential relation (not necessarily a lss), given by
$$K(N) \subseteq \mathrm{T}M \,.$$

Description as a vector field along a map

Aim of this work

Determine which linearly singular systems admit a **description in terms of a vector field along a map.**

This is advantageous because:

- A vector field along a map naturally defines **two differential relations**

$$D = \mathrm{T}\alpha^{-1}\big(K(N)\big) \subseteq \mathrm{T}N \qquad \text{and} \qquad D' = K(N) \subseteq \mathrm{T}M\,.$$

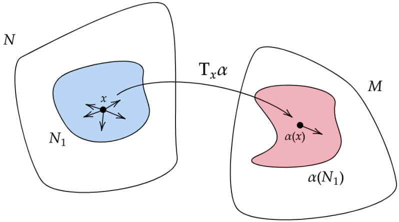
  which can be seen as a "Lagrangian" and a "Hamiltonian" formalism.
- Under suitable conditions, K can be used to **relate the solutions and constraint functions** of both systems.
- This formulation (known as the description in terms of an **evolution operator**) is useful for dealing with singular Lagrangian systems in both classical and contact mechanics.

Differential relation as a vector field along a map

Let $D \subset \mathrm{T}N$ and $\alpha \colon N \to M$, s.t. $N_1 = \tau_N(D)$ is a closed submanifold. If, for any $x \in N_1$ and $v_1, v_2 \in D_x$, we have

$$\mathrm{T}_x\alpha(v_1) = \mathrm{T}_x\alpha(v_2)\,.$$

Then $\mathrm{T}_x\alpha$ **collapses** each fibre D_x **to a single vector** in $\mathrm{T}_{\alpha(x)}M$.



So there exists a unique map (not necessarily smooth) $K \colon N_1 \to \mathrm{T}M$ along $\alpha|_{N_1}$ such that $\mathrm{T}_x\alpha(D_x) = K(x)$ for all $x \in N_1$.

Differential relation as a vector field along a map

This vector field along a map defines (under regularity assumptions) a linearly singular system, whose solutions are the paths $\xi\colon I \to N$ s.t.:

$$\mathrm{T}\alpha \circ \dot{\xi} = K \circ \xi\,.$$

Thus,

- We have three differential relations:

$$D \subset \mathrm{T}N\,, \qquad \mathrm{T}\alpha^{-1}(K(N_1)) \subseteq \mathrm{T}N\,, \qquad \mathrm{T}\alpha(D) = K(N_1) \subseteq \mathrm{T}M\,,$$

- Solutions of the first are solutions of the second. The converse is not true in general.
- Solutions of the first and second map to solutions of the third by composing with α. But, again, not every solution has this form.

Differential relation as a vector field along a map

Let $K\colon \mathbb{R}^2 \to \mathrm{T}\mathbb{R}$ be given by $K(x,y) = (x, xy)$ over $\alpha(x,y) = x$.
We have

$$K(\mathbb{R}^2) = \{(x,v) \in \mathrm{T}\mathbb{R} \mid x \neq 0 \text{ or } v = 0\},$$

Thus, $\eta(t) = t^2$ satisfies $\dot{\eta}(I) \subset K(\mathbb{R}^2)$.
However, solving $K(\xi(t)) = (\eta(t), \dot{\eta}(t))$ for a path $\xi(t) = (x(t), y(t))$ forces $x(t) = t^2$ and $y(t) = 2/t$ for $t \neq 0$. As $y(t)$ tends to infinity as $t \to 0$, the path $\xi(t)$ cannot be continuous.

Proposition

Let K be a vector field along a map α with **locally constant rank**, *and let $\eta\colon I \to M$ be a solution of the differential relation $K(N)$. Then,*

- *There locally exists a smooth path $\xi\colon I \to N$ s.t. $\dot{\eta} = K \circ \xi$.*
- *If, moreover, K is an injective immersion, ξ is unique.*
- *If, moreover, K is an embedding, then ξ is unique and exists globally.*

Linearly singular systems as vector fields along maps

Now, assume that D is described by a linearly singular system (A, b), so $D = A^{-1}(b(N))$. In this case,

$$\mathrm{T}_x\alpha(v_1 - v_2) = 0\,, \iff \mathrm{Ker}\, A_x \subseteq \mathrm{Ker}\, \mathrm{T}_x\alpha\,,$$

for any $v_1, v_2 \in D_x$ and $x \in N_1$.

If A has locally constant rank, this condition implies the existence of a **unique vector bundle map**

$$\psi\colon \mathrm{Im}\, A|_{N_1} \to \mathrm{T}M\,,$$

such that $\psi \circ A|_{N_1} = \mathrm{T}\alpha|_{N_1}$.
With this map, we can write the **vector field along** α explicitly, as

$$K = \psi \circ b|_{N_1}\,.$$

Linearly singular systems as vector fields along maps

If, moreover, $\operatorname{Ker} A = \operatorname{Ker} \mathrm{T}\alpha$, then ψ defines an **isomorphism** between $\operatorname{Im} A_x$ and $\operatorname{Im} \mathrm{T}_x\alpha$. And, in this case,

$$\xi \text{ solves } (A, b) \iff \xi \text{ solves the system defined by } K\,.$$

i.e.

$$A \circ \dot{\xi} = b \circ \xi \iff \mathrm{T}\alpha \circ \dot{\xi} = K \circ \xi\,.$$

In other words,

Theorem

Let (A, b) be a linearly singular system on N. If there exists a map $\alpha \colon N \to M$ such that

$$\operatorname{Ker} A = \operatorname{Ker} \mathrm{T}\alpha\,,$$

then the system (A, b) can be described in terms of a vector field along α.

Example: The evolution operator for classical mechanics

In classical mechanics, from a Lagrangian function $L \in \mathscr{C}^\infty(\mathrm{T}Q)$, one can define the following linearly singular system:

$$\begin{tikzcd} \mathrm{TT}Q \arrow[r, "(J{,}\widehat{\omega_L})"] \arrow[d] & \mathrm{TT}Q \oplus \mathrm{T}^*\mathrm{T}Q \\ \mathrm{T}Q \arrow[ur, Rightarrow, "(\Delta{,}\mathrm{d}E_L)"'] & \end{tikzcd}$$

where J is the vertical endomorphism, ω_L is the Lagrangian 2-form, Δ is the Liouville vector field, and $E_L = \Delta(L) - L$ is the Lagrangian energy.

The solutions $\xi\colon I \to \mathrm{T}Q$ to this linearly singular system

$$\widehat{\omega_L} \circ \dot{\xi} = \mathrm{d}E_L \circ \xi\,, \qquad J \circ \dot{\xi} = \Delta \circ \xi\,,$$

are the solutions to the Euler–Lagrange equations. The second equation is just the SODE condition.

Example: The evolution operator for classical mechanics

Now, if one considers the Legendre map

$$\begin{array}{rccc} \mathcal{F}L\colon & \mathrm{T}Q & \longrightarrow & \mathrm{T}^*Q \\ & (q,v) & \longmapsto & \left(q, \dfrac{\partial L}{\partial v}(q,v)\right) \end{array}$$

Then one can prove that

$$\operatorname{Ker}(J, \widehat{\omega_L}) = \operatorname{Ker} J \cap \operatorname{Ker} \widehat{\omega_L} = \operatorname{Ker} \mathrm{T}\mathcal{F}L \,,$$

and, hence, there exists a **vector field along the Legendre map**, $K\colon \mathrm{T}Q \to \mathrm{T}\mathrm{T}^*Q$, which provides a description of the Euler–Lagrange equations. In coordinates,

$$K(q,v) = v^i \frac{\partial}{\partial q^i}\bigg|_{\mathcal{F}L(q,v)} + \frac{\partial L}{\partial q^i} \frac{\partial}{\partial p_i}\bigg|_{\mathcal{F}L(q,v)} .$$

Conclusions

- We have identified the **conditions** under which a linearly singular system can be described with an **evolution operator**.
- This can be successfully **applied to recover** the vector fields along the Legendre map $\mathcal{F}L$ of both **classical mechanics and contact mechanics**.

What's next?

- Apply this formalism to **relate and classify** the **constraints** of Lagrangian and Hamiltonian formalisms,
- Find **descriptions** in terms of vector fields along maps for **other mechanical systems** (time-dependent, non-holonomic...).

References

- W. C. Rheinboldt, "Differential-algebraic systems as differential equations on manifolds", *Math. Comput.* **43** (1984), 473–482.
- G. Marmo, G. Mendella, W. M. Tulczyjew, "Symmetries and constants of the motion for dynamics in implicit form", *Ann. Inst. H. Poincaré Phys. Théor.* **57** (1992), 147–166.
- X. Gràcia and J. M. Pons, "A generalized geometric framework for constrained systems", *Differ. Geom. Appl.* **2** (1992), 223–247.
- X. Gràcia and J. M. Pons, "Symmetries and infinitesimal symmetries of singular differential equations", *J. Phys. A: Math. Gen.* **35** (2002), 5059–5077.
- X. Gràcia and J. M. Pons, "On an evolution operator connecting Lagrangian and Hamiltonian formalisms", *Lett. Math. Phys.* **17** (1989), 175–180.